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THE DESIGN OF A BEAM WITH CONTROLLED FORCE UNITS†

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The problem of choosing the optimum arrangement and forces of the actuators on a beam acted upon by a transverse load is considered. Using Green's function for the corresponding boundary-value problem, explicit formulae are obtained which express the solution of the problem as a function of the points of application of the actuators, after which the problem is replaced by its discrete analogue. In the final analysis, the problem reduces to a problem of mathematical programming, which is investigated and solved for different cases of the specification of the loads.

When designing structures with force units (actuators) which enable the structures to adapt to external loads, it is necessary to choose not only the optimal control of the actuators but, also, the number of such actuators and the positions at which they are placed [1]. Here, the force units are controlled by a local processor which is a part of the structure. The locations of the actuators and the instructions (programme) for the operation of the actuators are interrelated. The problem of designing the set of force units is solved below in the case of a beam. A special case of the problem has been considered previously, when the position and dimensions of a single actuator, occupying a section (of the piezoelastic cover plate type), were chosen [1].

1. FORMULATION OF THE PROBLEM

Consider an elastic beam of constant stiffness acted upon by a specified force $F(x)$. In addition to the force (x), actions, in the form of a force or moments of intensity $p(x)$ which are created by the actuators, can be applied to the beam.

The locations of the actuators and their forces are not specified in advance and only the constraints on these forces are given. It is required to find the locations of the actuators and their forces which minimize the deflection of the beam $u(x)$ when $x \in [0, 1]$. where $x = 0$ and $x = 1$ are the coordinates of the ends of the beam.

In Figs 1 and 2, diagrams, which explain the idea of force and moment actuators, are shown for a rigidly clamped beam (Fig. 1) and for cantilever beams (Fig. 2). A rigid body (the ground, a housing, etc.) is indicated by hatching. The force actuators are represented by the squares. The piezopatches, which are actuators of moments, are shown conventionally on the right-hand side of Fig. 2.

The moments produced by the force actuators are shown in the scheme on the left-hand side of Fig. 2. The moment on the ith pillar is $M_i = h(P_i - P_{i-1})$, where h is the height of the pillar and p_i is the force produced by the ith force actuator. After the moments M_i have been determined, we find the forces

$$P_i = P_{i-1} - M_i/h, \quad P_1 = -M_1/h$$

It is often easier to create pulling forces than pushing forces (a pushing force is created by the leftmost actuator in Fig. 2). By locating a symmetric set of actuators under the beam, it is only possible to create moments by applying pulling forces.

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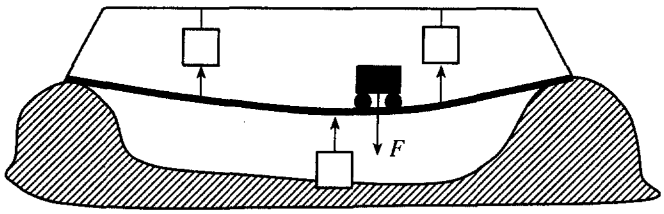

Fig. 1

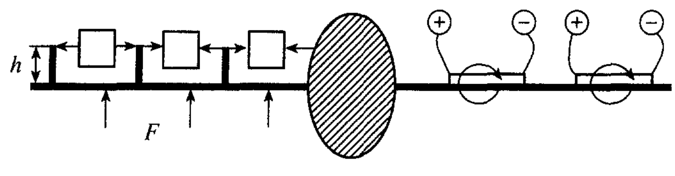

Fig. 2

It is required to determine where the actuators have to be replaced, what the magnitude of their forces must be and when which groups of actuators must be switched on (as a function of the magnitude and position of the external load at a given instant).

The locations of the actuators and their forces can be described by a single function $p(x)$ using the following rule: if $p(x) = 0$, there is no actuator at the point x and, if $p(x) \neq 0$, there is an actuator at the point x which acts with a force $p(x)$.

The deflection $u(x)$ of the beam under the action of the force $F(x)$ and the force actuators (force units which create additional force applied to the beam) of magnitude $p(x)$ is determined from the solution of the equation [2]

$$u^{\mathrm{IV}} = p + F \tag{1.1}$$

with the boundary conditions

$$u(0) = u(1) = 0, \quad u'(0) = u'(1) = 0 \tag{1.2}$$

(the condition of rigid clamping is considered here but other conditions can be used).

When moment actuators (force units which create additional moments in the beam) of magnitude $p(x)$ are used [1], we have the equation

$$(u'' + p)'' = F \tag{1.3}$$

with the same boundary conditions (1.2).

Here, the (constant) stiffness of the beam is taken to be equal to unity, which does not lead to any loss in generality of the treatment.

In the above formulation, the class A, to which the function $p(x)$ belongs, has still not been specified (that is, the type of actuators has not been specified). We will consider control forces $p(x)$ from the class (henceforth, summation is carried out from $j = 1$ to $j = m$)

$$\{\sum p_j \delta(x - y_j)\} \tag{1.4}$$

where $\delta(x)$ is the delta-function. This means that point actuators are used. Other classes of functions such as constants in sections, which are a model of piezopatches, can also be employed [1].

We take the constraint on the actuator forces in the form

$$a \le p_j \le b \tag{1.5}$$

that is, the actuators can create forces of any magnitude and sign in the range $[a, b]$.

It is required to minimize the deflection of the beam $u(x)$ along its length $[0, 1]$, that is, to solve the problem

$$\|u\|_{C[0,1]} \equiv \max_{x \in [0,1]} |u(x)| \to \min \tag{1.6}$$

The minimization is carried out using the values of $p(y)$ which satisfy condition (1.5) and, at the same time, $u(x)$ is determined by solving problem (1.1), (1.2) or (1.3), (1.2).

Remark. If there are no constraints on the forces generated by the actuators, problem (1.1), (1.2), (1.5) has the trivial solution $p(y) = -F(y)$, which corresponds to min $= 0$ in (1.6). When the constraints (1.5) or constraints associated with the class of controls A are present, the solution $p(x) = -F(x)$ may turn out to be impermissible. The requirement concerning the use of a finite number of actuators is a typical constraint which leads to the fact that the solution $p(x) = -F(x)$ will be impermissible. For example, the trivial solution will be impermissible in the case of the class of controls (1.4).

The solutions of problems (1.1), (1.2) ($k = 1$) and (1.3), (1.2) ($k = 2$) can be written in the form [3] (integration is henceforth carried out over the range [0, 1])

$$u(x) = \int L(x, y)F(y)dy + I_k(x), \quad I_1(x) = \int L(x, y)p(y)dy, \quad I_2(x) = \int M(x, y)p(y)dy \tag{1.7}$$

$L(x, y)$ and $M(x, y)$ are the fundamental solutions [3] of problems of the deflection of a beam under the action of a point force and a point moment, that is,

$$L^{\mathrm{IV}} = \delta(x-y), \quad M^{\mathrm{IV}} = -\delta''(x-y) \tag{1.8}$$

with boundary condition (1.2).

The functions $L(x, y)$ and $M(x, y)$ can be found in explicit form. They are third-degree polynomials which are "matched" at the point $x = y$ in accordance with the right-hand sides of Eqs (1.8). A simple computer program has been written for determining them.

Discretization of the problem with respect to the variable x. The use of the quantity $\|u\|_{C[0,1]}$ implies a consideration of $|u(x)|$ for a finite number of points. In order to avoid the problems which are associated with this, we will consider the maximum of $u(x)$ at a bounded number of points $\{x_1, \ldots, x_n\} \subset [0, 1]$, which we shall call points of deflection observation. Naturally, the question arises regarding the difference between $\|u\|_{C[0,1]}$ and $\|\mathbf{u}\| = \max_{i=1,\ldots,n}|u(x_1)|$.

The estimate

$$|u(x) - u(x_i)| \le (\|F\|_{C[0,1]} + \|p\|_{C[0,1]})|x - x_i|^4/4!$$

holds for the solution of Eq. (1.1).

If the points $\{x_1, \ldots, x_n\}$ are uniformly distributed in the range [0, 1] with a step size Δ, then

$$\left|\|u\|_{C[0,1]} - \|\mathbf{u}\|\right| \le (\|F\|_{[0,1]} + \|p\|_{[0,1]})\Delta^4/4!$$

The discretization with respect to the variable x, when the step size of the subdivision is reduced, is based on this.

The discrete problem. Suppose $\{y_1, \ldots, y_m\}$ are the points of a possible arrangement of the actuators. The solution of the two problems (1.7) for $p(y)$ from the class (1.4) takes the form

$$u(x_i) = G(x_i) + \sum L_{ij}p_j, \quad G(x) = \int L(x, y)F(y)dy \tag{1.9}$$

where $L_{ij} = L(x_i, y_j)$ for problem (1.1), (1.2), $L_{ij} = M(x_i, y_i)$ for problem (1.3), (1.2), $G(x)$ is a known function and p_j are the forces of the actuators at the point y_j ($p_j = 0$ means that there is no actuator at the point y_j).

The basis of the discretization with respect to the variable p will be given below.

Introducing the vectors

$$\begin{gathered} \mathbf{u} = \{u(x_i), i = 1, \ldots, n\} \in R^n, \quad \mathbf{y}_j = \{L_{ij}, i = 1, \ldots, n\} \in R^n, \\ \mathbf{y}_0 = \{G(x_i), i = 1, \ldots, n\} \in R^n \end{gathered} \tag{1.10}$$

the solution (1.9) can be written in the form

$$\mathbf{u} = \mathbf{y}_0 + \sum \mathbf{y}_j p_j \in R^n \tag{1.11}$$

The discretization of the problem corresponding to problem (1.6) has the form

$$\|\mathbf{u}\| \to \min \tag{1.12}$$

The vector $\mathbf{u}$ is specified by equality (1.11) and the values of p_j satisfy constraint (1.5).

2. THE SET OF POSSIBLE VALUES OF THE DEFLECTIONS AT THE POINTS OF OBSERVATION IN THE CASE OF CONSTRAINED ACTUATOR FORCES

Introducing the new variable q_j according to the rule

$$p_j = a + kq_j, \quad k = b - a, \quad q_j \in [0, 1],$$

instead of (1.11) we obtain a relation which differs from (1.11) in the free term and the normalizing factor k. We have

$$\mathbf{u} = \mathbf{Y}_0 + k\sum \mathbf{y}_j q_j \in R^n, \quad 0 \le q_j \le 1; \quad \mathbf{Y}_0 = \mathbf{y}_0 - \sum \mathbf{y}_j \tag{2.1}$$

The right-hand side of the first equation of (2.1) with the above-mentioned constraint on the values of q_j is a set kP which has been shifted by the vector $\mathbf{Y}_0$, where

$$P = \{\sum \mathbf{y}_j q_j: 0 \le q_j \le 1\} \tag{2.2}$$

To solve problem (1.12), (1.11), (1.5), it is therefore necessary to describe the set P (2.2).
We consider the points $\{\mathbf{y}_1, \dots, \mathbf{y}_m, \mathbf{y}_{m+1} = 0\}$ and form a finite number of sums

$$\sum \mathbf{y}_j s_j = \mathbf{Y}_s, \quad s = 1, \dots, N; \quad N = m + (m-1) + \dots + 1 = (m+1)m/2 \tag{2.3}$$

where s_j takes the values 0 or 1.

Assumption. The equality $P = \operatorname{conv}\{\mathbf{Y}_s, s = 1 \dots, N\}$ holds (conv denotes a "convex combination").
Actually, the assumption asserts that, for any point $\mathbf{y} \in P$, numbers $\{\mu_n\}$ and vectors $\{\mathbf{Y}_n\}$ are found such that

$$0 \le \mu_n \le 1, \quad \mu_1 + \dots + \mu_N = 1, \quad \mathbf{Y}_1\mu_1 + \dots + \mathbf{Y}_N\mu_N = \mathbf{y} \tag{2.4}$$

or, by virtue of the definition of the vectors $\mathbf{Y}_n$ (2.3)

$$\sum \mathbf{y}_j^1 s_j^1 \mu_1 + \dots + \sum \mathbf{y}_j^N s_j^N \mu_N = \mathbf{y} \tag{2.5}$$

Equality (2.5) will be satisfied if, for any $0 \le q_j \le 1$, the problem

$$s_j^1\mu_1 + \dots + s_j^N\mu_N = q_j \tag{2.6}$$

with the condition 2.4 is solvable. We recall that s_j^n takes the values 0 and 1, and that $\{\mu_n\}$ satisfy conditions (2.4). The sums $\mathbf{s}^1\mu_1 + \dots + \mathbf{s}^N\mu_N$, where the components of the vector $\mathbf{s}^n = (s_1^n, \dots, s_N^n)$, s_j^n take the values 0 or 1, with condition (2.4), specify a unit hypercube in R^m [4]. The vector $\mathbf{q}$ with the coordinates q_j, $0 \le q_j \le 1$ belongs to the unit cube, and, consequently, problem (2.4), (2.6) is solvable.
Note that $P = \operatorname{conv}\{\mathbf{Y}_s, s = 1, \dots, N\}$ is a polyhedron.

Corollary. By virtue of the assumption and formula (2.1), the polyhedron $K = \mathbf{Y}_0 + kP$ can be written in the following form. We form the finite sums $\Sigma \mathbf{y}_j s_j = \mathbf{Z}_s$ ($s = 1, \dots, N$), where s_j take the values a or b. Then

$$K = \operatorname{conv}\{\mathbf{Z}_s, s = 1, \dots, N\} \tag{2.7}$$

3. SOLUTION OF THE PROBLEM IN THE CASE OF A CONSTANT EXTERNAL LOAD

Taking into account the results in Section 2, we conclude that problem (1.11), (1.12), (1.5) is equivalent to the following problem

$$\|\mathbf{u}\| \to \min, \quad \mathbf{u} \in K \tag{3.1}$$

that is, the problem reduces to minimizing the function $\|\mathbf{u}\|$ in the polyhedron K (2.7). An attempt can be made to solve this problem using the methods of optimization theory. However, it can be solved in a much more simpler manner if a geometrical analysis of the set of possible deflections of the beam K and of the set $\|\mathbf{u}\| \le c$ is carried out.

The polyhedron K. Consider the sums $\Sigma \mathbf{y}_j p_j$ in which R of the quantities p_j are non-zero and the remaining $m - R$ of the quantities p_j are zero. These sums form the polyhedra

$$P_R = \operatorname{conv}\{\textstyle\sum \mathbf{y}_j s_j;\ s_j = 0 \quad \text{or} \quad s_j = 1\}$$

and, moreover, only those $\mathbf{y}_j$ for which $p_j \neq 0$ occur in a sum, that is, each polyhedron P_R corresponds to the operation of R actuators. We shall call these polyhedra "polyhedra of level R". The equality $P = \cup P_R$ holds. Suppose a point $\mathbf{u} \in P$. Then, it belongs to a certain polyhedron P_R, that is, the displacements $\mathbf{u}$ at the points of observation can be realized using the R actuator corresponding to the polyhedron P_R.

The polyhedron K and the cube $D(c)$ touch a certain point $\mathbf{x}_c$. As a result, we arrive at the problem

$$\mathbf{Y}_1\mu_1 + \ldots + \mathbf{Y}_N\mu_N = \mathbf{x}_c, \quad 0 \le \mu_n \le 1, \quad \mu_1 + \ldots + \mu_N = 1 \tag{3.2}$$

This is a so-called problem of convex combinations [5]. A method for solving it has been presented earlier in [5].

We note that, in the case being considered, the set $\{q_j\}$ is of interest, that is, the solution of problem (2.1), rather than $\{\mu_n\}$ which is the solution of problem (3.2). In order to obtain the solution of problem (2.1), we recall that any vector $\mathbf{Y}_s$ is a sum of the form $\Sigma \mathbf{y}_j s_j$. Then, if a vector $\mathbf{y}_j$ occurs with non-zero coefficients s_j in the sums forming the vectors $\mathbf{Y}_{s(1)}, \ldots, \mathbf{Y}_{s(p)}$, then $q_j = \mu_{s(1)} + \ldots + \mu_{s(p)}$.

The condition $\|\mathbf{u}\| \le c$. This condition (by virtue of the definition of $\|\mathbf{u}\|$ (3.1)) defines a cube $D(c)$ with edges $x_i = \pm c$. As the parameter c increases from zero to infinity, this cube increases from a point (the origin of the coordinate system when $c = 0$) to infinity (when $c \to \infty$).

It is now possible to solve problem (3.1).

Value of the minimum in problem (3.1). Two cases are possible: $0 \notin K$, when $\min_{\mathbf{u} \in K}\|\mathbf{u}\| = c > 0$ and $0 \in K$, when $\min_{\mathbf{u} \in K}\|\mathbf{u}\| = 0$.

In the case when $0 \notin K$, when $c = 0$, the polyhedron $D(c)$ coincides with the origin of the coordinate system. We shall increase the value of the parameter c, and, at the same time, the cube $D(c)$ will be enlarged. If the polyhedron K does not contain the origin of coordinates, then, at a certain value of the parameter c^*, the enlarging cube $D(c)$ touches K (Fig. 3). This gives the solution of problem (3.1): the value c^*, at which the first touching occurs, is the value of the minimum in problem (3.1).

In the case when $0 \in K$, zero values can be assigned to the deflections at the points of observation. In other respects, the solution is similar to that described above.

4. BASIS OF THE DISCRETIZATION OF THE PROBLEM WITH RESPECT TO THE VARIABLE p

Consider expression (1.7) in the case when the discretization with respect to x has been carried out and that with respect to p has still not been carried out. At the points of observation of the deflections, we obtain the equality

$$\begin{aligned}
&\mathbf{u} = \mathbf{y}_0 + \int \mathbf{L}(y)p(y)dy \\
&\mathbf{u} = \{u(x_i), i = 1, \ldots, n\} \in R^n, \quad \mathbf{L}(y) = \{L(x_i, y), i = 1, \ldots, n\} \in R^n, \\
&\mathbf{y}_0 = \{G(x_i), i = 1, \ldots, n\} \in R^n
\end{aligned} \tag{4.1}$$

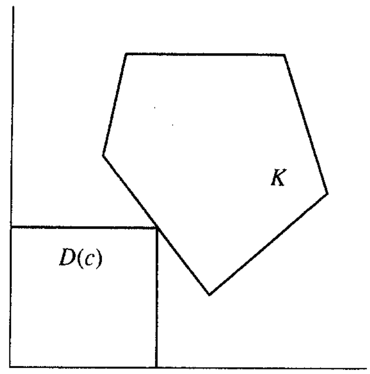

Fig. 3

We introduce the pair of lines $\Gamma = \{\mathbf{x} = a\mathbf{L}(y), \mathbf{x} = b\mathbf{L}(y) : y \in [0, 1]$. The integral on the right-hand side of equality (4.1) with conditions (1.4) and (1.5) defines a cone $P_d = \mathrm{conv}\{\Gamma, 0\}$ with base $P_c^0 = \mathrm{conv}\Gamma$ and vertex at zero. This follows from the corollary presented in Section 2 and the fact that the convex shell of any set is formed by convex combinations of a finite number of points of this set [4]. Equality (1.11) defines a cone $K_c = P_c + \mathbf{y}_0$ (the cone P_c, shifted by the vector $\mathbf{y}_0$) with base $K_c^0 = P_c^0 + \mathbf{y}_0$.

The discretization used in Section 1 is an approximation of the line Γ by a broken line with vertices at the points $\{\mathbf{Y}_s \in \Gamma\}$ and the approximation of the curvilinear cones by the polyhedral cones K and P. The quality of the approximation improves as the number of points increases.

We will now give a general characteristic of the method being used. Expression (1.7) can be considered as an integral functional $F{:}A \to C([0, 1])$. Then, the set of deflections of the beam under the action of actuators of intensity $p \in A$ is $F(A)$, the image of the set A in the case of the mapping F. The discretization carried out above corresponds to the approximation of the sets A and $F(A)$ by finite sets. At the same time, it turns out that the finite dimensional approximation $F(A)$ is quite easily calculated in an explicit form as a convex combination of known points.

5. A VARIABLE EXTERNAL LOAD. THE CHOICE OF THE LOCATION OF THE SYSTEM OF ACTUATORS, THEIR FORCES,ORDER OF THE SWITCHING ON OF GROUPS OF ACTUATORS AND INSTRUCTION TO THE BEAM PROCESSOR

We will consider the case when the external load depends on a parameter: $F = F(x, t), x \in [0, 1]$, and $t \in L = [0, T]$ is the parameter. The external load, which the actuators attempt to "compensate", was fixed above. The effective position of the actuators and their forces were uniquely determined. If the external load is variable, then its own set of actuators for each load is necessary to "compensate" it. If the load can take many different values, then the straightforward application of the theory from the preceding section can lead to the requirement that a large number of actuators are used. We shall show that the problem can be solved using a restricted set of actuators.

Proceeding as above, we arrive at the expressions

$$u(x, t) = \int L(x, y)F(y, t)dy + \int L(x, y)p(y, t)dy \tag{5.1}$$

$$u(x, t) = \int L(x, y)F(y, t)dy + \int M(x, y)p(y, t)dy \tag{5.2}$$

which differ from (1.7) in that they depend on the parameter t. From the start, the external load $F(y, t)$ depends on t and, by virtue of this, both the deflections $u(x, t)$ and the controls $p(y, t)$ also become dependent on t. Next, we transfer from (5.1) or (5.2) to the problem (compare with problem (3.1))

$$\|\mathbf{u}(t)\| \to \min, \quad \mathbf{u} \in K(t), \quad t \in [0, T] \tag{5.3}$$

where

$$K(t) = \mathrm{conv}\{\mathbf{Y}_s, s = 1, \ldots, N\} + \mathbf{Y}_0(t) \in R^n \tag{5.4}$$

Note that, in expression (5.4), the vector $\mathbf{Y}_0(t)$ only depends on t (see Section 2 for the definition of the vectors $\{\mathbf{Y}_s, s = 1, \dots, N\}$ and $\mathbf{Y}_0$ and the number N). This is important in the further treatment.

Two control methods are possible: (1) problem (5.3), (5.4) is solved for each value of the parameter t; (2) problems (5.3), (5.4) are first solved for different $t \in [0, T]$ and the solution obtained is subsequently used for a given t. The systems being considered are sometimes referred to as "smart" systems. In these terms, it is possible to say that two levels of "smartness" of a system can be distinguished.

1. *A "universal" system.* The system has a large number of actuators or actuators can be placed in specified positions and, by solving problem (5.3), (5.4), it calculates which actuators have to be switched on and their forces, starting from the condition $D(c) \cap K(t) \neq \varnothing$. In this case, the system can be adjusted (if this is possible in general) under any external load.
2. *A system acting according to the "if–then" principle.* The system switches on or switches off actuators according to the rule: if ***<current value of the load parameter>*** then ***<switch on group of actuators from the list>***. This system can only be adjusted under an active external load from a known class.

The first case is not achieved in practice in pure form. We shall therefore dwell on the second case.

We select a number c such that $D(c) \cap K(t) \neq \varnothing$ for all $t \in L = [0, T]$, which is always possible. In this case, when $t \in L$, the polyhedron $K(t)$ describes a certain trajectory in R^n which always has common points with the cube $D(c)$. It is necessary to find these points, for which it suffices to indicate the set of points $\mathbf{x}_1, \dots, \mathbf{x}_M$ in the polyhedron $K(0)$ which are such that just one of the points $\mathbf{x}_i + \mathbf{y}_0(t)$ lies in $D(c)$ for any value of the loading parameter t. In addition, it is desirable that there should be as few such points as possible.

We will now explain this requirement. The point at $K(0)$ corresponds to a system of actuators: The systems of actuators, corresponding to the points $\mathbf{x}_1, \dots, \mathbf{x}_M$, solve the problem (actually, when the external load is such that the point $\mathbf{x}_i + \mathbf{y}_0(t)$ belongs to $D(c)$, it is necessary to switch on the system of actuators corresponding to the point $\mathbf{x}_i$). From a practical point of view, it would be desirable to have fewer actuators. The solution of the problem within the framework of an *"if–then"* control is possible by virtue of the fact that, during the continuous motion of the polyhedron $K(t)$, its subsets P_K corresponding to the systems of actuators, intersect the cube $D(c)$ at values of t from a certain range $[t_a, t_b]$. A finite number of such ranges $[t_a, t_b]$ covers the whole of $L = [0, T]$.

The following algorithm is proposed for choosing the points.

0. We find the values L_0 of the loading parameter t for certain $\mathbf{y}_0(t) \in D(c)$. This case corresponds to the fact that, without the action of the actuators, the deflection of the beam does not exceed a value c. The set L/L_0 of values of the parameter remains.

1. We consider a point $\mathbf{x}_c$ at which the polyhedron $K(t)$ touches the cube $D(c)$ (the worst version). It is necessary to have the system of actuators corresponding to this case. We find the values L_c of the loading parameter t for which $\mathbf{x}_c \in D(c)$. If there are several such points $\mathbf{x}_c$, we repeat the procedure for each points. As a result $L \backslash L_0 \backslash L_c$ remains. We then operate with iterations which can depend on our choice of points in the polyhedron (see below).

2. Accompany the exit of the points of contact $\mathbf{x}_c$ from $D(c)$, the subsets of the polyhedron $K(t)$ enter the cube $D(c)$. The polyhedron $K(0)$ is subdivided into subsets P_K. We choose P_K, starting out from the following conditions:

(a) K is the smallest possible number;

(b) $\{P_K + \mathbf{y}_0(t)\} \cap D(c) \neq \varnothing$ for the greatest possible number of values of the parameter t.

3. We repeat stage 2 until we have exhausted all the values of the parameter t.

It can be seen that stage 0 and 1 of the algorithm are uniquely determined, but stage 2 admits of a certain arbitrariness. As the result of the application of the algorithm, we obtain a finite number of schemes from which it is possible to choose the preferable scheme.

Sensors and the processor in the system. If the system is controlled according to the rule: *if $t \in L_c$, then* ***<switch on a group of actuators from the list>***, it is necessary to take care determining the loading parameter t. It would be convenient to determine the current loading parameter t in terms of the current characteristics of the deformation of the beam (the displacements etc.). It needs to be kept in mind that control actions p can be applied to the beam simultaneously with an external load $F(x, t)$. Suppose there is a functional $\Phi(u, p)$ such that $t = \Phi(u(x, t), p(t))$. By measuring (using sensors, for example) the current value of $u(x, t)$ and calculating the value of the functional Φ, we can determine the value of the loading parameter t.

The calculations and switching of the actuators described above require the existence of a processor which generates these actions. These operations are quite simple and, therefore, the processor does not need to have a higher power.

The form of the functional Φ is substantially associated with the specific class of loading. For a specific class of loadings, it is fairly simple to construct the functional Φ, but it is not convenient to give a general method for its construction here.

"Knowledge" in the system. The set of instructions of the *"if–then"* form is treated as knowledge [6]. As we see, knowledge is a part of a "smart" structure. A "Universal" system does not lock knowledge but it can be inefficient for purposes of operational control, since it solves a rather large problem. A system of the *"if–then"* type uses simple instructions and can be fast acting even with a low-power processor. In this connection, it is suitable for purposes of operational control. The following separation of the functions of the systems of an equal level of "smartness" mentioned above is possible. The *"if–then"* system accomplishes operational control of a structure in situations which are described in these instructions. The "Universal" system, while not participating directly in the operational control, carries out instructions for the *"if–then"* system. As a rule, the operational control system is a part of the structure (with the exception of cases of remote control). The system for carrying out the instructions can be both a part of the structure as well as an external system. Analogues of the systems which have been described and techniques for their distribution in biological and social systems can be imagined.

6. EXAMPLE

The design of a "smart" structure consists of two components: the structure itself (the type, number and siting of the actuators) and the program (instructions for the control of the actuators) which is introduced into the built-in processor of the structure as a computer program.

The example considered below is carried out, in particular, with the aim of showing that, in the proposed method, the structural and program parts of the design are not simply interrelated (a statement concerning their "interelatedness" has become a common topic for papers on "smart" structures, see [7, 8] and the bibliography these) and they are determined as parts of the solution of the same problem.

We will consider a beam corresponding to the range $[0, 1]$ of the x axis. An external point load (Fig. 1) moves along the beam between the points $x = 0$ and $x = 0.5$. We shall observe the deflections at the points $x_1 = 0.25$ and $x_2 = 0.5$. We will give the possible locations of the actuators A_1, A_2, A_3 at the points $y_1 = 0.25, y_2 = 0.5$ and $y_3 = 0.75$.

We take the constraints on the forces of the actuators (1.5) in the form $0 \le p_j \le 1$ $(a = 0, b = 1)$.

The evolution of the polyhedron $K(t)$ in the case of the problem being considered is shown in Fig. 4. The parameter $t = 0.05i$ $(i = 0, \dots, 10)$ corresponds to an external load, which moves from one end of the beam to its centre in steps of 0.05 (the positions of the cone $K(t)$ for $i = 0, 5, 8, 9, 10$ are shown in Fig. 4). In the case under consideration, the cube $D(c)$ converts into a square, since the problem is two-dimensional (all points of observation of the deflection are taken). The square $D(c)$ was found as the smallest square which has an intersection with all of the polyhedrons $K = K(t)$, $t = 0.05i$ $(i = 0, \dots, 10)$.

Determination of the optimal set of actuators. The set of actuators and the positions in which they are placed are determined from a qualitative analysis of Fig. 4. It is necessary to establish precisely which "subpolyhedrons" of the polyhedron $K(t) = 1457638$ are intersected by the square $D(c)$ (the small square at the origin of coordinates in Fig. 4). In Fig. 4, point 8 corresponds to the deflections of the uncontrolled beam at the points $x_1 = 0.25$ and $x_2 = 0.5$. The segments 18, 28 and 38 (of the level 1 polyhedron) correspond to the operation of a single actuator A_1, A_2, A_3 respectively, and the parallelograms 3158, 8142 and 8263 (of the level 2 polyhedron) correspond to the operation of the pairs of actuators $A_1 + A_3, A_1 + A_2, A_2 + A_3$ respectively. The remaining part of the polyhedron $K(t)$, which, in this case, is the parallelogram 2476, corresponds to the operation of all three actuators.

When $i \le 5$, minimum deflections can be maintained by the single actuator A_1 (at the point $y_1 = 0.25$) and its force increases as the point of application of the force $F(t)$ advances to the middle of the beam. Subsequently, when $i = 6, 7, 8$, the actuator A_1 can be used (at the point $y_1 = 0.25$) together with the actuator A_2 (at the point $y_2 = 0.5$) or the actuator A_3 (at the point $y_3 = 0.75$). When $i = 9$, A_1 can be used together with A_2 (but A_1 and A_3 can no longer ensure optimal control) or all three actuators can be used. When $i = 10$, the single actuator A_2 can be used. In the final analysis, we arrive at the conclusion that the minimum number of actuators is equal to two and these will be the actuators A_1 and A_2 (located at the points $y_1 = 0.25$ and $y_3 = 0.75$). Note that the use of the single actuator A_2 (at the middle of the beam) does not provide the best solution. The action of a single actuator with a different force is

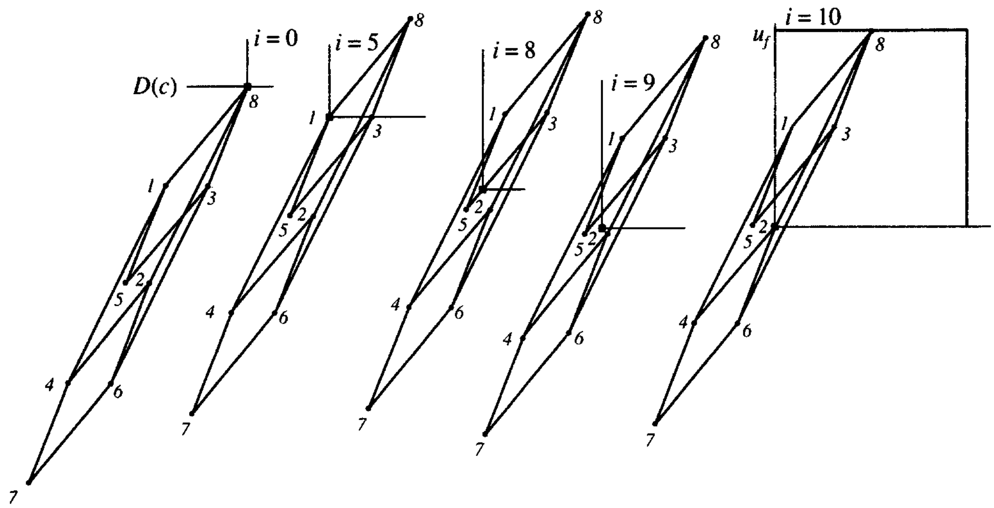

Fig. 4

represented by the line 82. Except for the cases when $i = 0$ and $i = 1, i = 10$, this line did not intersect the square $D(c)$, that is, the single actuator A_2 cannot accomplish optimal control.

The maximum deflection of the controlled beam occurs when $i = 3$ (the external force is applied at the point $y = 0.15$). In the case of the uncontrolled beam, the maximum deflection u_f occurs when the force is applied at the point 0.5. The relative decrease in the deflection due to the control $u_f/c = 78$ (see Fig. 4 when $i = 10$, point 8 corresponds to deflections of the uncontrolled beam).

The development of instructions for the operation of the actuators. The instructions for the operation of the actuators are obtained from a quantitative analysis of Fig. 4. We shall use the actuators A_1 and A_2. In order to write down the instructions, it is necessary to find the coefficients of the convex combinations of vectors, specifying a point which belongs to the square $D(c)$. These coefficients will be equal to the forces p_1 and p_2, which the actuators A_1 and A_2 must develop, respectively. The values of p_1 and p_2 are shown in Table 1 as a function of the coordinate of the point at which the force F $(0.05i)$ is applied.

Table 1

i	0	1	2	3	4	5	6	7	8	9	10
p_1	0	0.05	0.2	0.47	0.68	1	1	0.84	0.63	0.31	0
p_2	0	0	0	0	0	0	0.16	0.44	0.5	0.69	1

Note that, except for the case when $i = 3$, the forces p_1 and p_2 are not determined uniquely. The differences in the determination of the quantities p_1 and p_2 are proportional to the size of the square $D(c)$.

7. A CANTILEVER BEAM. EXAMPLE

Suppose a force (a uniform pressure) $F(x, t) = F(t)$, which is constant along the axis of the beam, is applied to the cantilever beam shown in Fig. 2. We consider the problem with the choice of the points of observation: $x_1 = 0.5$ and $x_2 = 1$ (the middle and the free end of the beam) and the points for the possible siting of the actuators: $y_1 = 0.25, y_2 = 0.5$ and $y_3 = 0.75$. The boundary conditions have the form

$$u(0) = u'(0) = 0, \quad u''(1) = u'''(1) = 0$$

(the rigid clamping is at $x = 0$ and the free end is at $x = 1$).

Control is carried out by means of moment actuators. In this case, the fundamental solution is determined from the equation

$$N^{\mathrm{IV}} = \delta'(x-y)$$

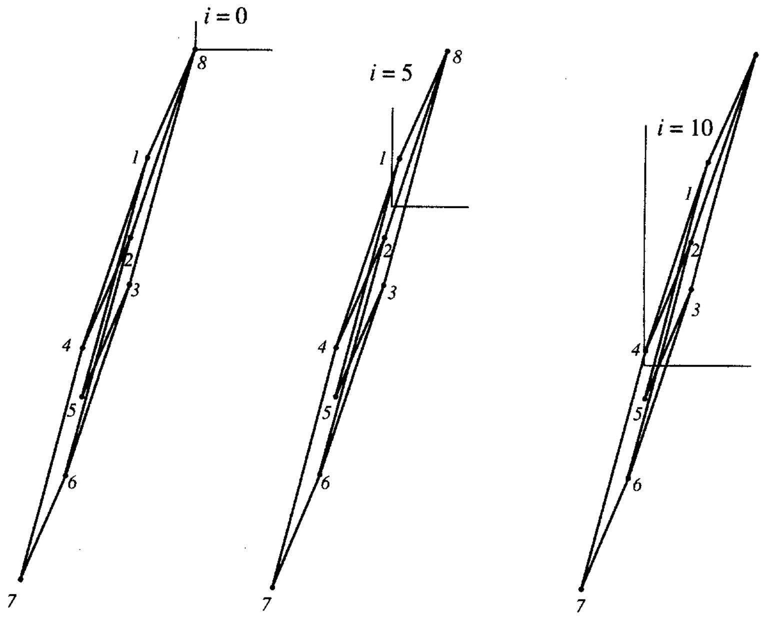

Fig. 5

We take the constraints on the force of the actuators (1.5) in the form $0 \le p_j \le 1$ ($a = 0, b = 1$).

The evolution of the polyhedron $K(t)$ when $F(x, t) = t/2$, $t = i = 0, \ldots, 10$ (the data for $i = 0, 5, 10$ are presented) is shown in Fig. 5. In the case being considered, the polyhedron $K(t)$ contains the point 0 for all $i = 0, \ldots, 10$. Consequently, it is possible to achieve zero deflections at the points of observation for all loads. It followed from an analysis of the complete set of diagrams, which are partially shown in Fig. 5, that, when $i \le 8$, it is possible to use just the actuators A_1 (at the point $y_2 = 0.25$) and A_2 (at the point $y_2 = 0.5$). When $i > 8$, it is necessary to use all three actuators.

8. ACTUATORS OF THE 0–1 TYPE

We will now consider the case when the actuators only have two states: "switched on" (the force $p_j = 1$) and "switched off" (the force $p_j = 0$). A continuous analogue of this problem is considered in Section 6. In this case, the problem takes the form

$$\|\mathbf{u}\| \to \min, \quad \mathbf{u} \in K(t); \quad K(t) = \mathbf{y}_0(t) + \sum \mathbf{y}_j s_j \in R^n \tag{8.1}$$

where s_j take the values zero and unity.

The minimum in (8.1) is equal to

$$c = \max_{t \in L} \min_{\mathbf{u}} \|\mathbf{u}\|$$

The problem is solved by finding the smallest square $D(c)$ for which the condition

$$D(c) \cap K(t) \ne \varnothing, \quad \forall t \in L = [0, T]$$

is satisfied.

It is necessary to order the points of the set $K(t)$ with respect to the quantity $\sum s_j$ (which is equal to the number of non-zero terms in the sum (8.1), that is, the number of actuators required to realize the corresponding state). This ordering enables one to satisfy the requirements of stage 2a of the design algorithm.

Several stages in the evolution of the points $K(t)$ are shown in Fig. 6 for the case considered in Section 6 for actuators of the 0–1 type. Point 8 corresponds to deflections of the uncontrolled beam. Points 1, 2, and 3 (rank 1) correspond to the operation of a single actuator A_1, A_2, A_3 respectively, Points *5*, *4*, and *6* (rank 2) correspond to the operation of a pair of actuators $A_1 + A_3$, $A_1 + A_2$, $A_2 + A_3$ respectively. Point 7 (rank 3) corresponds to the operation of all three actuators.

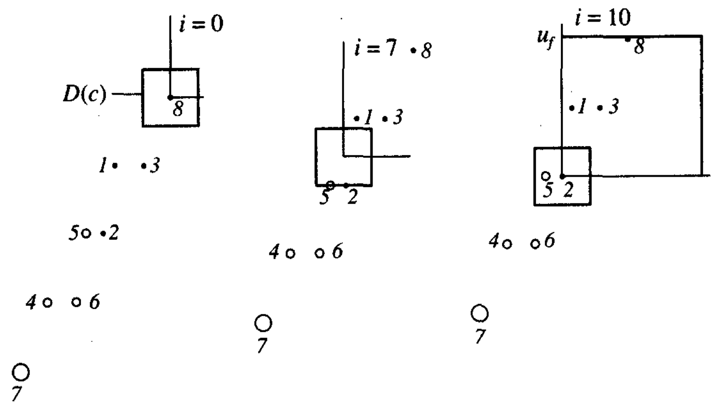

Fig. 6

The size of the square $D(c)$ in the problem being considered is determined by the case $i = 7$ in Fig. 6. The scheme for the switching of the actuators is determined by the passage of the system of points $K(t)$ across $D(c)$. It follows from an analysis of the complete pattern ($i = 0, \ldots, 10$) that, when $i \leq 3$, there is no need to use the actuators. When $i = 4$, it is necessary to use the actuator A_1 or A_2, when $i = 5, 6$, the actuator A_1, and, when $i = 7, 8, 9, 10$, the actuator A_2 or A_1 and A_3. The ratio $u_f/c = 4.8$, see Fig. 6 when $i = 10$.

The use of controls of the 0–1 type forms the basis of methods of "discrete" optimization [9]. The number of possible states of a system with a control of the 0–1 type is comparatively small, which enables one to solve the problem using a computer by direct inspection (in combination with standard methods for investigating functions for an extremum). A comparison of the relative decreases in the deflection: 4.8 for discrete optimization and 78 for a continuous control (see Section 6), is indicative of the low efficiency of "discrete" optimization compared with continuous optimization. The examples in Sections 6 and 8 show that the continuous and discrete designs can also be distinguished in their constructional part (the different distribution sites of the actuators).

In the case of continuous designs, the use of inspection to solve the problem is impossible. In the example from Section 6, the solution was obtained for $n = 3$ possible positions of the actuators in the case of $i = 10$ positions of the external load, and a calculation of the forces of the actuators with an accuracy of 1/100. The solution of the problem by inspection would require an investigation of the function for an extremum $100^n I = 10^7$ times and the solution of this problem using a single-processor computer would take several hours. In the case of a 0–1 design, $2^n I = 80$ versions would have to be looked through. When $n = 6$, the solution of the continuous problem by the method of inspection becomes practically impossible.

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